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IDEAL MATRICES. III

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Dedicated to the memory of Ernst G. Straus

In this paper ideal matrices with respect to ideals in the maximal order of an algebraic number field are connected with the different of the field and with group matrices in the case of normal fields whose maximal order has a normal basis.

1. Let F be an algebraic number field of degree n over Q. Let $\mathfrak{O}$ be the maximal order of F and $\mathfrak{a}$ an ideal in $\mathfrak{O}$. Let $\omega_1,\dots,\omega_n$ be a Z-basis of $\mathfrak{O}$ and $\alpha_1,\dots,\alpha_n$ a Z-basis of $\mathfrak{a}$. Then there exists an integral $n \times n$ matrix A such that

$$A\begin{pmatrix}\omega_1\\ \vdots\\ \omega_n\end{pmatrix} = \begin{pmatrix}\alpha_1\\ \vdots\\ \alpha_n\end{pmatrix}. \tag{1}$$

A is called an ideal matrix for $\mathfrak{a}$. The ideal defines a representation module for $\mathfrak{O}$. A change of bases replaces A by UAV where U, V are unimodular $n \times n$ Z-matrices and conversely, to any U, V there exist bases for $\mathfrak{O}$ and $\mathfrak{a}$ such that UAV is an ideal matrix for $\mathfrak{a}$.

2. THEOREM 1. *Let (j) denote the family of embeddings of F in the complex numbers. Then, suppressing the superscript* (1),

$$A\begin{pmatrix}\omega_1 & \omega_1^{(2)} & \cdots & \omega_1^{(n)}\\ \vdots & \vdots & & \vdots\\ \omega_n & \omega_n^{(2)} & \cdots & \omega_n^{(n)}\end{pmatrix} = \begin{pmatrix}\alpha_1 & \alpha_1^{(2)} & \cdots & \alpha_1^{(n)}\\ \vdots & \vdots & & \vdots\\ \alpha_n & \alpha_n^{(2)} & \cdots & \alpha_n^{(n)}\end{pmatrix}. \tag{2}$$

Hence we have

THEOREM 2. $A = (\alpha_i^{(j)})(\omega_i^{(j)})^{-1}$.

Both theorems have trivial proofs. Let α be any integral generator of F. Then there exists an integral matrix X such that $\det(\omega_i^{(j)})\det X = \pm$ different α. Further, $(\omega_i^{(j)})(\omega_i^{(j)})'$ is the so-called discriminant or trace matrix of $\mathfrak{O}$ where $'$ stands for the transpose and the determinant of the above matrix product is the discriminant of the maximal order.

REMARK. In an earlier paper by this author and M. Newman, Comm. Pure Appl. Math. **9** (1956), 85–91, the matrix $(\omega_i^{(j)})$ was called the discriminant matrix. This term is at present being used by other authors for the product of the matrix and its transpose.

3. The case of a normal field F.

THEOREM 3. *The matrix* $(\omega_i^{(j)})$ *is a group matrix in the sense of* [**1**] *provided that the* ω_i*'s are a normal basis for* $\mathfrak{O}$.

Proof. For a suitable ordering of the elements $\sigma_1 = 1, \sigma_2, \ldots, \sigma_n$, of the Galois group G of F the following equation can be made to hold

$$\omega_i^{(j)} = \omega_i^{\sigma_j^{-1}}. \tag{3}$$

If $\mathfrak{a}$ too has a normal basis then A is the product of two group matrices.

4. Ideal matrices for products of ideals and for the conjugate ideal.

THEOREM 4. (*The so-called AUB theorem, see* [Taussky **3**] *Theorem* 1): *Let A, B be ideal matrices corresponding to the ideals $\mathfrak{a}$, $\mathfrak{b}$ in $\mathfrak{O}$. Then there exists an integral unimodular matrix U such that AUB is an ideal matrix for the product $\mathfrak{a}\mathfrak{b}$.*

THEOREM 5. *The matrix equation* (2) *for the ideal* $\mathfrak{a}^{(j)}$ *is obtained immediately since the matrix* $(\alpha_i^{(j)})$ *consists of columns of conjugates.*

5. The ideal matrix and Stickelberger theory for F a field generated by a root of unity ζ_m, see e.g. [Ireland and Rosen, [**4**] page 246]. In the case under consideration every prime ideal not containing m, raised to a certain symbolic power coming from the integral group ring of the Galois group of F is a principal ideal. From the tools developed in 4 the corresponding idea can be applied to A.

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